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Multi-component spin model on a Cayley tree

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Abstract. A general spin model on the Cayley tree lattice, which includes both the q component Potts and the Ashkin–Teller models, is considered. The free energy in zero field is evaluated in a closed form and found to be analytic in temperature. The model exhibits no long-range order in the sense that the probability of finding two sites far away to be in spin states α and β is a constant, independent of α and β.

We also evaluate the susceptibility per site, χ_R, for a region R in the centre of the lattice, defined to be the summation of the site–site correlations between R and the whole lattice L. For the linear size of R to be any finite fraction of that of L, χ_R diverges at the Bethe–Peierls temperature(s) T_{BP}, while for $R \equiv L$, χ_R diverges at temperature(s) different from T_{BP}.

1. Introduction

The spin-$\frac{1}{2}$ Ising model on a Bethe lattice has been of renewed recent interest. The model was first introduced some twenty years ago by Kurata *et al* (1953) who obtained a closed-form expression for its free energy. It was recognized only recently (Eggarter 1974, von Heimburg and Thomas 1974, Matsuda 1974) that, while its free energy is analytic in temperature, the model actually possesses a phase transition characterized by a divergent susceptibility (and a zero long-range order). As this kind of critical behaviour is not without physical interest (see e.g. Stanley and Kaplan 1966), it is useful to extend the consideration to other models. The lattice gas of hard molecules on a Bethe lattice has been considered by Runnels (1967). We study in this paper a general spin model which includes both the Potts (1952) and the Ashkin–Teller (AT) (1943) models. Our discussion uses the Perron–Frobenius theorem and is more direct and applicable to the general Cayley tree lattice. Critical behaviour similar to that of the spin-$\frac{1}{2}$ Ising model on the Bethe lattice is obtained.

The outline of our paper is as follows. In § 2, the general spin model is defined and the partition function evaluated in a closed form. In § 3, we establish the absence of a long-range order by evaluating the appropriate correlation function. Both of these results are valid for Cayley tree lattices. The susceptibility is evaluated in § 4 for a Bethe lattice, and is found to diverge in certain temperature ranges. By defining the susceptibility slightly differently which neglects the surface effect, the temperature ranges in which the susceptibility diverges are found to be different. These latter temperatures are shown in § 5 to be related to the Bethe–Peierls temperatures of the spin model.

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2. The partition function

Following Essam and Fisher (1970), we use the term Cayley tree to denote a connected graph which contains no cycles. A Bethe lattice is then a Cayley tree having the same valence at all vertices. Generally, there are $N-1$ edges in a Cayley tree of N vertices. The vertices of valence 1 are said to be on the 'surface'. An example of a Cayley tree which has 12 surface vertices is shown in figure 1. Note that there exists a unique path between any two vertices of a Cayley tree.

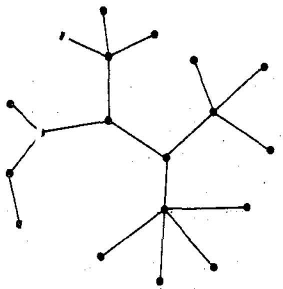

Figure 1. A Cayley tree with 12 surface vertices.

Consider a q component spin system on a Cayley tree with spin variables $\xi_i = 1, 2, \ldots, q$; $i = 1, 2, \ldots, N$. The Hamiltonian can be quite generally written as

$$\mathscr{H} = -\sum_{\langle ij \rangle} J(\xi_i, \xi_j) \tag{1}$$

where $\langle ij \rangle$ stands for all nearest-neighbouring pairs, and $-J(\xi_i, \xi_j) = -J(\xi_j, \xi_i)$ is the interaction energy between two sites in states ξ_i and ξ_j. The partition function is then

$$Z = \sum_{\{\xi\}} \prod_{\langle ij \rangle} u(\xi_i, \xi_j) \tag{2}$$

where

$$u(\xi, \xi') = \exp[\beta J(\xi, \xi')] \qquad \beta = (kT)^{-1}. \tag{3}$$

The free energy per site is

$$f = -kT \lim_{N \to \infty} N^{-1} \ln Z. \tag{4}$$

We shall restrict our considerations to the spin model satisfying the relation

$$\sum_{\xi'=1}^{q} u(\xi, \xi') = w \tag{5}$$

independent of ξ. Two examples are the q component (standard) Potts and the AT models for which

$$J(\xi, \xi') = \epsilon \delta_{Kr}(\xi, \xi') \qquad \text{(Potts)}$$

$$J(\xi, \xi') = \begin{bmatrix} \epsilon_0 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & \epsilon_0 & \epsilon_3 & \epsilon_2 \\ \epsilon_2 & \epsilon_3 & \epsilon_0 & \epsilon_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 & \epsilon_0 \end{bmatrix} \qquad \text{(AT)} \tag{6}$$

and

$$w = e^{\beta\epsilon} + q - 1 \qquad \text{(Potts)}$$

$$w = u_0 + u_1 + u_2 + u_3 \qquad \text{(AT)} \tag{7}$$

where $u_i = \exp(\beta\epsilon_i)$. For $\epsilon > 0$, the Potts model is a generalization of the ferromagnetic Ising model ($q = 2$) which has a q fold degenerate ground state. The $\epsilon < 0$ Potts model, on the other hand, has a highly degenerate ground state and has been termed the 'orthogonal' model by Alexander and Yuval (1974). Another example is the planar (vector) Potts model with

$$J(\xi, \xi') = \epsilon \cos[2\pi(\xi - \xi')/q] \tag{8}$$

which reduces to the classical Heisenberg model in the limit of $q \to \infty$.

In the Ising case, the partition sum (2) has been evaluated by means of high temperature expansion (Kurata *et al* 1953) and spin variable transformation (Eggarter 1974). The summation can actually be carried out simply and more generally for the general spin model (5) as follows. We start from a spin ξ_s on the surface of the Cayley tree and observe that its summation yields a factor w and reduces Z into $Z = wZ'$. Here Z' is the partition function of the Cayley tree with the spin ξ_s and its associated edge deleted. The process can obviously be continued. After eliminating all but one spins and all the edges, we arrive at

$$Z = w^{N-1} \sum_{\xi=1}^{q} 1 = qw^{N-1}. \tag{9}$$

The free energy per site in the thermodynamic limit is then

$$f = -kT \ln w \tag{10}$$

which is analytic in the temperature T.

To reveal the non-analyticity in the free energy, we shall in later discussions introduce a field $-h$ to one of the spin components and consider more generally the free energy $f = f(T, h)$ in both variables T and h. While (10) shows $f(T, 0)$ analytic in T, it will be seen that $f(T, h)$ can be non-analytic in h.

3. The correlation function and the absence of long-range order

The method of summation described above can also be used to evaluate the correlation function for the spin model (5). For an appropriate definition of the correlation function, consider first the probability $P_l(\alpha, \beta)$ of finding two vertices l steps apart in the respective spin states α and β. Let the two spin sites be A and B. We have

$$P_l(\alpha, \beta)=\langle\delta_{\mathrm{Kr}}(\xi_{\mathrm{A}}, \alpha)\delta_{\mathrm{Kr}}(\xi_{\mathrm{B}}, \beta)\rangle = Z^{-1} \sum_{\{\xi\}}{}' \prod_{\langle ij\rangle} u(\xi_i, \xi_j) \tag{11}$$

where $\langle\ \rangle$ denotes the thermal average and the prime in the summation denotes that the spin states at A and B are fixed at $\xi_{\mathrm{A}}=\alpha$, $\xi_{\mathrm{B}}=\beta$. Note that the usual spin–spin correlation for the spin-$\frac{1}{2}$ Ising model ($\xi_i=\sigma_i$) is given in terms of the probabilities as

$$\langle\sigma_0\sigma_l\rangle = \sum_{\sigma_{\mathrm{A}}\sigma_{\mathrm{B}}} \sigma_{\mathrm{A}}\sigma_{\mathrm{B}}P_l(\sigma_{\mathrm{A}}, \sigma_{\mathrm{B}}).$$

Since we expect $\lim_{l\to\infty} P_l(\alpha, \beta)$ to be independent of α and β when there exists no correlation between the spins, the site–site correlation function can be taken to be

$$\Gamma_l(\alpha, \beta)=P_l(\alpha, \beta)-q^{-2}. \tag{12}$$

There is no long-range order if $\lim_{l\to\infty}\Gamma_l(\alpha, \beta)=0$, which says that the probability of finding two sites far away to be in states α and β is a constant.

To evaluate the correlation function $\Gamma_l(\alpha, \beta)$, we proceed as in § 2. Starting from the surface vertices of a Cayley tree, we can carry out the spin sums one by one and eliminate all vertices and edges except those lying on the unique path between A and B (the full lines in figure 2). Number these vertices $1, 2, \ldots, l-1$ running from A to B.

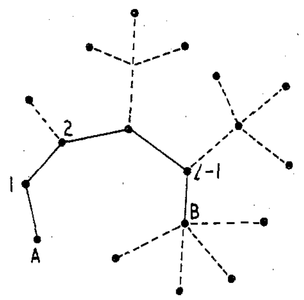

Figure 2. The full lines denote the unique path between two vertices A and B of a Cayley tree.

We then obtain

$$\begin{aligned} P_l(\alpha, \beta) &= Z^{-1}w^{N-(l+1)} \sum_{\xi_1\ldots\xi_{l-1}} u(\alpha, \xi_1)u(\xi_1, \xi_2)\ldots u(\xi_{l-1}, \beta) \\ &= q^{-1}[\mathbf{V}^l]_{\alpha\beta} \end{aligned} \tag{13}$$

where we have used (9) and $\mathbf{V}$ is a $q\times q$ matrix whose elements are

$$V_{\alpha\beta}\equiv[\mathbf{V}]_{\alpha\beta}=w^{-1}u(\alpha, \beta). \tag{14}$$

Let the eigenvalues of $\mathbf{V}$ be λ_j and the αth component of the (normalized) eigenvector associated with λ_j be $\phi_j(\alpha)$. Then we may write

$$[\mathbf{V}^l]_{\alpha\beta} = \sum_{j=1}^{q} \lambda_j^l\phi_j(\alpha)\phi_j^*(\beta). \tag{15}$$

It is easy to see that, as a consequence of (5), $\lambda_1 \equiv 1$ is an eigenvalue of $\mathbf{V}$ with $\phi_1(\alpha) = 1/\sqrt{q}$ for all α. It follows then

$$\Gamma_l(\alpha, \beta) = q^{-1} \sum_{j=2}^{q} \lambda_j^l \phi_j(\alpha) \phi_j^*(\beta). \tag{16}$$

Now the matrix $\mathbf{V}$ has strictly positive elements and $\phi_1(\alpha)$ is positive. Then by the Perron–Frobenius theorem, we know that $\lambda_1 = 1$ is in fact the largest eigenvalue and is nondegenerate. Hence we find

$$\lim_{l \to \infty} \Gamma_l(\alpha, \beta) = 0 \tag{17}$$

which establishes the absence of a long-range order.

For the Potts and AT models, where $\mathbf{V}$ is a cyclic or doubly cyclic matrix, we then find for the Potts models:

$$\phi_j(\alpha) = q^{-1/2} \exp[2\pi \mathrm{i}(j-1)(\alpha-1)/q] \tag{18}$$

$$\lambda_1 = 1$$

$$\lambda_2 = \lambda_3 = \ldots = \lambda_q = 1 - q/w \qquad \text{(standard Potts)} \tag{19a}$$

$$\lambda_j = \frac{1}{w} \sum_{k=1}^{q} \exp[\beta\epsilon \cos(2\pi k/q) + 2\pi \mathrm{i} k(j-1)/q]$$

$$\to I_{j-1}(\beta\epsilon)/I_0(\beta\epsilon) \qquad \text{as } q \to \infty \qquad \text{(planar Potts).} \tag{19b}$$

For the AT model:

$$\phi_j(\alpha) = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{20}$$

$$\lambda_1 = 1$$

$$\lambda_2 = (u_0 + u_1 - u_2 - u_3)/w$$

$$\lambda_3 = (u_0 - u_1 + u_2 - u_3)/w$$

$$\lambda_4 = (u_0 - u_1 - u_2 + u_3)/w. \tag{21}$$

We also find

$$\Gamma_l(\alpha, \alpha) = q^{-2} \sum_{j=2}^{q} \lambda_j^l. \tag{22}$$

4. Susceptibility and fluctuation

To investigate the non-analytic behaviour of the free energy $f(T, h)$, where $-h$ is the field introduced to the spin component α, we consider the zero-field susceptibility

$$\chi = \lim_{N \to \infty} N^{-1} (\langle n_\alpha^2 \rangle - \langle n_\alpha \rangle^2), \tag{23}$$

where

$$n_\alpha = \sum_i \delta_{\mathrm{Kr}}(\xi_i, \alpha). \tag{24}$$

In view of the absence of a long-range order, we shall take

$$\langle n_\alpha \rangle = N/q. \tag{25}$$

It follows then from (11) and (12)

$$\chi = \lim_{N\to\infty} N^{-1} \sum_{r,s} \Gamma_{l(r,s)}(\xi_r = \alpha, \xi_s = \alpha). \tag{26}$$

Here, $l(r, s)$ is the number of steps between sites r and s. One may recognize that (26) is the usual fluctuation relation.

To carry out the summations in (26), we now consider a regular Bethe lattice. The lattice, shown in figure 3, is composed of $g+1$ generations of lattice points and has a

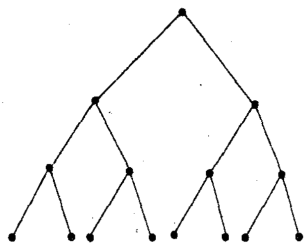

Figure 3. A four-generation ($g = 3$) Bethe lattice with coordination number $\gamma = 3$ ($B = 2$).

coordination number $\gamma = B+1$. Number the generations by $n(=0, 1, \ldots, g)$ so that there are B^n vertices in the nth generation. We may rewrite (26) for the $g+1$ generation lattice as

$$\chi^{(g)} = N_g^{-1} \sum_{n=0}^{g} \sum_{n'=0}^{g} \chi_{nn'} \tag{27}$$

where

$$N_g = (B^{g+1} - 1)/(B-1) \tag{28}$$

and

$$\chi_{nn'} = \sum_{r\in n} \sum_{s\in n'} \Gamma_{l(r,s)}(\xi_r = \alpha, \xi_s = \alpha) \tag{29}$$

is the correlation between the generations n and n'. Finally, after taking the thermodynamic limit $g\to\infty$, we obtain

$$\chi = \lim_{g\to\infty} \chi^{(g)}. \tag{30}$$

The expression (29) for $\chi_{nn'}$ can be explicitly evaluated using (16). To illustrate, consider the Potts model whose correlation function $\Gamma_l(\alpha, \alpha)$ has been given in (22). We find from (27)

$$\chi_{nn'}^{(\mathrm{Potts})} = q^{-1}(1-q^{-1}) \sum_{r\in n} \sum_{s\in n'} \lambda^{l(r,s)} = q^{-1}(1-q^{-1}) \sum_{l=|n-n'|}^{n+n'} G(l)\lambda^l \tag{31}$$

where

$$\lambda = (e^{\beta\epsilon} - 1)/(e^{\beta\epsilon} + q - 1) \tag{32}$$

and $G(l)$ is the number of l step paths between vertices in n and n'.

Without loss of generality, we need only to consider $n \geqslant n'$. It is clear that, starting from a given vertex A in n, there is precisely one vertex in n' which is $n - n'$ steps away. Since there are B^n vertices in n, we find

$$G(n - n') = B^n. \tag{33}$$

Similarly, there are $B - 1$ vertices in n' which are $n - n' + 2$ steps away from A, and generally $(B-1)B^{k-1}$ vertices in n' which are $n - n' + 2k$ steps away from A. Thus, for $n \geqslant n'$

$$\begin{aligned} q^2(q-1)^{-1}\chi_{nn'}^{(\text{Potts})} &= B^n\left(\lambda^{n-n'} + \sum_{k=1}^{n'} (B-1)B^{k-1}\lambda^{n-n'+2k}\right) \\ &= B^{n'}(B\lambda)^{n-n'}\{1 - \lambda^2[1 + (B-1)(B\lambda^2)^{n'}]\}(1 - B\lambda^2)^{-1}. \end{aligned} \tag{34}$$

For $n' \geqslant n$, we simply interchange n and n' in (34).

Substituting (34) and (28) into (27) and (30), and dropping terms that vanish in the thermodynamic limit, we find

$$q^2(q-1)^{-1}\chi = (1+\lambda)/(1-B\lambda) - \lim_{g\to\infty} H(g, \lambda) \tag{35}$$

where

$$H(g,\lambda) = [\lambda(B-1)/(1-B\lambda)(1-B\lambda^2)][1 + \lambda + \lambda(B-1)(B\lambda^2)^g/(1-B\lambda)]. \tag{36}$$

It is now clear that

$$\chi = \begin{cases} q^{-1}(1-q^{-1})(1+\lambda)^2/(1-B\lambda^2) & B\lambda^2 \leqslant 1 \\ \infty & B\lambda^2 \geqslant 1. \end{cases} \tag{37}$$

The susceptibility therefore diverges for $B\lambda^2 \geqslant 1$. Using (32), the condition $B\lambda^2 \geqslant 1$ implies

$$T \leqslant T_c(\sqrt{B}) \tag{38}$$

where

$$T_c(x) = \begin{cases} \epsilon/k \ln[(q+x-1)/(x-1)] & \epsilon \geqslant 0 \\ |\epsilon|/k \ln[(x-1)/(x+1-q)] & \epsilon \leqslant 0, \sqrt{B}+1 > q. \end{cases} \tag{39}$$

A striking result is that χ diverges in the orthogonal model ($\epsilon < 0$) provided that $\sqrt{B}+1 > q$. This appears to be a unique property of the Bethe lattice, for it is known that for an Ising antiferromagnet, the $q = 2$ orthogonal model, on a square or cubic lattice ($B = 3$ or 5), the free energy $f(T, h)$ is analytic in h at sufficiently low temperatures (Brascamp and Kunz 1973).

The result (37) for χ can be readily generalized to the general spin model (5). Using (16) and (27), we find

$$\chi^{(g)} = \sum_{k=2}^{q} \chi_k^{(g)} \tag{40}$$

where

$$\chi_k^{(g)} = N_g^{-1} \sum_n \sum_{n'} \chi_{nn',k} \tag{41}$$

$$\chi_{nn',k} = q^{-1}|\phi_k(\alpha)|^2 \sum_{r\in n} \sum_{s\in n'} \lambda_k^{l(r,s)}. \tag{42}$$

The summations in (42) can be carried out as in (31) with λ_k in place of λ. We then find

$$\chi = \begin{cases} q^{-1} \sum_{k=2}^{q} |\phi_k(\alpha)|^2(1+\lambda_k)^2/(1-B\lambda_k^2) & B\lambda_k^2 \leqslant 1 \\ \infty & \text{any } B\lambda_k^2 \geqslant 1. \end{cases} \tag{43}$$

Thus, the susceptibility diverges whenever any

$$|\lambda_k| \geqslant 1/\sqrt{B}. \tag{44}$$

The above result serves to establish that the free energy $f(T, h)$ fails to be analytic in h, in at least the temperature ranges (44). For an Ising ferromagnet, it has been known (Müller-Hartmann and Zittartz 1974) more generally that the higher field derivatives of the free energy diverge in wider temperature ranges. The result is that $f(T, h)$ is nonanalytic in h for $T \leqslant T_{BP}$, where T_{BP} is the Bethe–Peierls temperature of the Ising model. A similar analysis can presumably be carried out for the present problem. We shall, however, proceed in another direction.

The effect on the critical behaviour due to the large number of surface vertices of the Bethe lattice appears to have been first observed by Runnels (1967). In the present problem the non-analyticity of $f(T, h)$ at T_{BP} manifests itself if one focuses attention to a central region of the lattice (Eggarter 1974). Proceeding along this line, we define the susceptibility for a central region R of a Bethe lattice L as the following generalization of (26):

$$\chi_R = N_R^{-1} \sum_{r\in R} \sum_{s\in L} \Gamma_{l(r,s)}(\xi_r = \alpha, \xi_s = \alpha) \tag{45}$$

where N_R is the number of sites in R. As we shall see, χ_R diverges at T_{BP} for any $R \neq L$.

Specify the region R by the index $0 < \nu < 1$ such that

$$R = \{n | 0 \leqslant n \leqslant \nu g\} \tag{46}$$

and

$$N_R = (B^{\nu g+1} - 1)/(B-1). \tag{47}$$

Equation (27) now reads

$$\chi^{(g)}(\nu) = N_R^{-1} \sum_{n=0}^{g} \sum_{n'=0}^{\nu g} \chi_{nn'}. \tag{48}$$

The last expression can again be evaluated using (29) and (16). For the Potts model, for example, we obtain the following in place of (35),

$$q^2(q-1)^{-1}\chi(\nu) = (1+\lambda)/(1-B\lambda) - \lim_{g\to\infty} (B\lambda)^{(1-\nu)g} H(g, \lambda). \tag{49}$$

Equation (49) is the same as (35) for $\nu = 1$. It is clear, however, that we have for all $\nu < 1$

$$\chi(\nu) = \begin{cases} q^{-2}(q-1)(1+\lambda)/(1-B\lambda) & |B\lambda| < 1 \\ \text{divergent} & |B\lambda| \geqslant 1. \end{cases} \tag{50}$$

Thus, the susceptibility $\chi(\nu)$ diverges for $|B\lambda|\geqslant 1$ or

$$T\leqslant T_c(B), \tag{51}$$

where $T_c(B)$ lies above $T_c(\sqrt{B})$. Similarly, the susceptibility $\chi(\nu)$ for the general spin model (5) diverges whenever any $|\lambda_k|\geqslant 1/B$. We shall show in § 5 that, indeed, these region(s) of divergences are related to the Bethe–Peierls temperature(s) of the spin system.

5. The Bethe–Peierls temperature

In this section, we determine the Bethe–Peierls critical temperature for the spin model (5).

Consider a lattice of coordination number γ and focus our attention to a particular site, A. Let $P_{\{n\}}(i)$, where

$$\{n\}=(n_1, n_2, \ldots, n_q)$$
$$n_1+n_2+\ldots+n_q=\gamma, \tag{52}$$

be the probability of finding A in the ith $(=1,\ldots,q)$ state and n_j of its γ neighbouring sites in the jth state. The probability that A is in the ith spin state is then

$$P_A(i)=\sum_{\{n\}} P_{\{n\}}(i). \tag{53}$$

Similarly, the probability that one of the γ neighbours of A is in the ith state is

$$P_B(i)=\gamma^{-1}\sum_j\sum_{\{n\}} n_i P_{\{n\}}(j). \tag{54}$$

For the system in a translationally invariant state, we then expect

$$P_A(i)=P_B(i) \qquad i=1,2,\ldots,q. \tag{55}$$

Only $q-1$ of the q equations in (55) are independent, since the summations over i on both sides of (55) are identically equal to 1.

The Bethe–Peierls approximation is to write (see, e.g., Huang 1963)

$$P_{\{n\}}(i)=FC^{\gamma}_{n_1\ldots n_q}\prod_{j=1}^{q}(u_{ij}z_j)^{n_j} \tag{56}$$

where

$$C^{\gamma}_{n_1\ldots n_q}=\gamma!/n_1!\ldots n_q! \tag{57}$$

is the multinomial coefficient, $u_{ij}\equiv u(i,j)$ is the Boltzmann factor (3), and z_j a *quasi-fugacity* introduced to represent the effect of the rest of the lattice. Equations (55) are then used to determine $z_1,\ldots,z_q$. The constant F in (56) is determined by the normalization

$$\sum_i\sum_{\{n\}} P_{\{n\}}(i)=1. \tag{58}$$

This leads to

$$F^{-1}=\sum_i f_i^{\gamma} \tag{59}$$

with

$$f_i=\sum_j u_{ij}z_j. \tag{60}$$

Using (56) and (59), we find

$$P_A(i)=f_i^\gamma$$
$$P_B(i)=F\gamma^{-1}z_i\frac{\partial}{\partial z_i}(F^{-1}). \tag{61}$$

Equations (55) now lead to

$$G_i(z_1,\ldots,z_q)=0 \qquad i=1,\ldots,q \tag{62}$$

where

$$G_i(z_1,\ldots,z_q)\equiv f_i^\gamma-z_i\sum_j u_{ij}f_j^{\gamma-1}. \tag{63}$$

Equations (62) are homogeneous in $z_1^\gamma,\ldots,z_q^\gamma$. The existence of a non-trivial solution is guaranteed because only $q-1$ of the q equations are independent.

Without loss of generality, we may take $z_1=1$ and consider $q-1$ of the q equations in (62), say, $i=2,\ldots,q$. Using (5), it is immediately seen that one solution to (62) is

$$z_2=z_3=\ldots=z_q=1 \tag{64}$$

which is valid at all temperatures. The other solutions, if any, are temperature dependent.

If another solution to (62) exists, we say that a transition occurs (under the Bethe–Peierls approximation) at the temperature the solution first appears. This is the Bethe–Peierls temperature T_{BP}.

To determine T_{BP} in the present problem, we expand $G_i(1,z_2,\ldots,z_q)$ near (64) and rewrite the $q-1$ equations as

$$\sum_{j=2}^{q}(z_j-1)G_{ij}=0 \qquad i=2,\ldots,q \tag{65}$$

where

$$G_{ij}=w^{-\gamma}(\partial G_i/\partial z_j)_{z_k=1}. \tag{66}$$

The condition that (65) has a nontrivial solution is

$$\det|G_{ij}|=0. \tag{67}$$

This is now the equation which determines T_{BP}. Note that $\det|G_{ij}|$ is a $(q-1)\times(q-1)$ determinant.

Using (63) and (15), it is easily found that

$$G_{ij}=\gamma V_{ij}-\delta_{ij}-(\gamma-1)[\mathbf{V}^2]_{ij}$$
$$=\sum_{k=1}^{q}\phi_k(i)(1-\lambda_k)[(\gamma-1)\lambda_k-1]\phi_k^*(j). \tag{68}$$

The summation in (68) actually runs from $k=2$ to q since $\lambda_1=1$. It is then convenient to introduce $\boldsymbol{\phi}$, the $(q-1)\times(q-1)$ matrix whose elements are $\phi_k(i)$ and $\boldsymbol{\Lambda}$, the diagonal matrix whose elements are $(1-\lambda_k)[(\gamma-1)\lambda_k-1]$, $i, k=2, \ldots, q$. Equation (68) then leads to

$$\det|G_{ij}| = \det(\boldsymbol{\phi}\boldsymbol{\Lambda}\boldsymbol{\phi}^{\dagger}) = \det(\boldsymbol{\phi}^{\dagger}\boldsymbol{\phi}) \det \boldsymbol{\Lambda} \tag{69}$$

where $\boldsymbol{\phi}^{\dagger}$ is the adjoint of $\boldsymbol{\phi}$. The first factor in (69) can be evaluated using the orthonormal relation

$$q^{-1} + \sum_{k=2}^{q} \phi_k(i)\phi_k^*(j) = \delta_{ij} \tag{70}$$

which leads to the value q^{-1}. T_{BP} is now determined by

$$q^{-1} \prod_{k=2}^{q} (1-\lambda_k)[(\gamma-1)\lambda_k - 1] = 0. \tag{71}$$

Now $\lambda_k < 1$ for $k \neq 1$. Hence the solutions are

$$B\lambda_k = (\gamma-1)\lambda_k = 1 \qquad k=2, \ldots, q. \tag{72}$$

We have obtained in § 4 the result that the susceptibility $\chi(\nu)$ diverges for $|B\lambda_k| \geqslant 1$. While for positive λ_k the temperature $|B\lambda_k| = 1$ is indeed the T_{BP} given by (72), for $\lambda_k < 0$ this is not the case. Now, as $\lambda_k < 0$ only for some special energy parameters, such as the orthogonal ($\epsilon < 0$) Potts model, it seems that by rewriting the condition (55) appropriately, one should be able to derive other T_{BP} which may lead to $B\lambda_k = -1$. An example is the $q=2$ orthogonal Potts model. In this model the ordered state is 'antiferromagnetic'; in place of (55), one writes

$$P_{\mathrm{A}}(1) = P_{\mathrm{B}}(2) \tag{73}$$

which indeed leads to the T_{BP} given by

$$B\lambda = -1. \tag{74}$$

(This T_{BP} (for $q=2$) happens to be the same as that of $\epsilon > 0$.) We have been unable, however, to extend the considerations in the most general case.

6. Summary

We have considered a general spin model on a Cayley tree. The free energy in zero field is analytic in temperature and there is no long-range order. Considered as a function of an external field h, the free energy is non-analytic at $h=0$ in certain temperature ranges. It is shown that when the surface effects are appropriately excluded, these temperatures coincide with the Bethe–Peierls temperatures of the spin model. Our results imply, in particular, that the free energy of an Ising antiferromagnet is nonanalytic at $h=0$ at low temperatures. This behaviour is different from that of the Ising antiferromagnet on a square or cubic lattice.

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Note added in proof. Properties of the spin correlation functions of the Ising system on a Cayley tree have been given recently by Falk.

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